



8 Linear maps and linear operators





Introduction

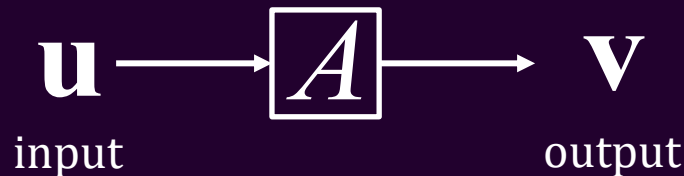
- In this topic, we will
 - Define maps and operators on vector spaces
 - Give some background
 - Define the image and range of maps
 - Give some examples





Background

- In engineering, any signal,
be it continuous-time or discrete-time,
may be represented as a vector
 - Voltages
 - Sound
 - Temperatures
 - Concentrations of chemicals
 - Global positioning system coordinates
- Engineers must extract information from such signals





Background

- We may be calculating, for example:
 - Average values of a signal or vector
 - Reducing noise within a signal
 - Increasing the resolution
 - Displaying an analog TV signal on a 4K screen
- In all cases, the output is another vector
 - In the first, the output is $\mathbf{R}^1$
 - In the second, it is the same vector space
 - In the third, it is a higher-dimensional vector space





Maps between vector spaces

- Given two vector spaces $\mathcal{U}$ and $\mathcal{V}$,
a map A is a function that takes a vector in one vector space
and *maps* it to a vector in another vector space

$$\mathcal{U} \longrightarrow \boxed{A} \longrightarrow \mathcal{V}$$

- We will write this as $A: \mathcal{U} \rightarrow \mathcal{V}$
 - Here, A must map every vector in $\mathcal{U}$ to some vector in $\mathcal{V}$
 - We will say $\mathbf{u} \in \mathcal{U}$ is mapped to the vector $A\mathbf{u} \in \mathcal{V}$
 - We may sometimes write $A(\mathbf{u})$ to avoid confusion,
 - For example $A(\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2)$





Definitions

- Given a map $A: \mathcal{U} \rightarrow \mathcal{V}$,
 - we will refer to $\mathcal{U}$ as the *domain*
 - we will refer to $\mathcal{V}$ as the *codomain*
- If $A: \mathcal{U} \rightarrow \mathcal{U}$, that is, the domain and codomain are equal,
 - we will refer to A as an *operator*
- If $\mathbf{u} \in \mathcal{U}$, then $A\mathbf{u} \in \mathcal{V}$ is the *image of $\mathbf{u}$ under the map A*
- If $\mathbf{v} \in \mathcal{V}$, and $\mathbf{u} \in \mathcal{U}$ such that $\mathbf{v} = A\mathbf{u}$,
 - we will say $\mathbf{u}$ is a *pre-image of $\mathbf{v}$* under the map A
- If $S \subseteq \mathcal{U}$, then AS is the collection of all $A\mathbf{u}$ where $\mathbf{u} \in S$
 - AS is the image of S under the map A
- Let $A\mathcal{U}$ be the collection of all images $A\mathbf{u}$ for every $\mathbf{u} \in \mathcal{U}$
 - This is said to be the *range of A*





Example

- Here are examples of two operators, each mapping $\mathbf{R}^3$ to $\mathbf{R}^3$
 - The first maps to cylindrical coordinates
 - The second to spherical coordinates

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{x^2 + y^2} \\ \tan^{-1}\left(\frac{y}{x}\right) \\ z \end{pmatrix} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{x^2 + y^2 + z^2} \\ \tan^{-1}\left(\frac{y}{x}\right) \\ \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \end{pmatrix}$$





Example

- Here are three other operators
 - The first maps the vector to the average value
 - The second subtracts off the average of the entries
 - The third averages each point with its neighbors

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \frac{x+y+z}{3} \end{pmatrix} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x - \frac{x+y+z}{3} \\ y - \frac{x+y+z}{3} \\ z - \frac{x+y+z}{3} \end{pmatrix} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \frac{x+y}{2} \\ \frac{x+y+z}{3} \\ \frac{y+z}{2} \end{pmatrix}$$





Summary

- Following this topic, you now
 - Understand the of a map or operator
 - A map takes vectors in the domain and maps them to vectors in the codomain
 - Know the terms image and range
 - We can take the image of a vector, a subset of the domain, or the entire domain
 - Have seen some examples





References

- [1] [https://en.wikipedia.org/wiki/Map_\(mathematics\)](https://en.wikipedia.org/wiki/Map_(mathematics))
- [2] https://en.wikipedia.org/wiki/Domain_of_a_function
- [3] <https://en.wikipedia.org/wiki/Codomain>
- [4] [https://en.wikipedia.org/wiki/Image_\(mathematics\)](https://en.wikipedia.org/wiki/Image_(mathematics))





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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